Vectors - A more in-depth understanding

Hamish McDonald

Note: This short write-up assumes a basic knowledge of vectors and some set theory. It is intended to introduce some of the more abstract maths underpinning vectors.

Section 0: Notation and some definitions

- We recall that a set in mathematics is just a collection of objects. These objects can be numbers, variables, or quite literally anything you like. Sets are denoted using *set brackets* $\{\}$. For example, consider the following set S.

$$S = \{1, 2, 3, 4, 5\}$$

- We recall that the symbol $\in$ is used to show something is an **element** of a particular set. For example, we could say $2 \in S$.
- Vectors are often written in maths papers with boldface. For example, $\mathbf{v}$.

Section 1: What is a vector, really?

Formally, a vector is any element of a **vector space** V. A **vector space** is just a set of objects that follow a specific set of rules (see the section below). The vectors that we are most familiar with are the vectors of the vector spaces $\mathbb{R}^n$, whose elements are n-tuples over the real numbers, $\mathbb{R}$.

$$(a_1, a_2, ..., a_n) \text{ where } a_i \in \mathbb{R} \text{ for all } i = 1, 2, ..., n$$

For example, the 2-tuple (ordered pair) $(2, 3)$ is an element of the vector space $\mathbb{R}^2$.

It is important to note that all sorts of things form vector spaces. For example, the set of all $n \times n$ matrices form their own vector space. So, in the context of that vector space, matrices are also vectors as well.

Vector spaces are defined *over a field* F. A field is a set of numbers that follow their own specific set of rules. The most common field used with vectors is that of the real numbers $\mathbb{R}$. Elements of this field, in the context of a vector space, are known as **scalars**.

Vector spaces also have an element $\mathbf{0} \in V$ named *zero* (think the ordered pair (0,0) in $\mathbb{R}^2$).

Vector spaces define a **binary operation** and a **binary function**. A binary operation is an operation that takes two elements from a set and returns another element from that set. A binary function is just a function that takes in two inputs and returns a single output.

The binary operation for vector spaces is *vector addition*, which assigns any two vectors $\mathbf{v}$ and $\mathbf{w}$ to a third vector in V written as $\mathbf{v} + \mathbf{w}$. For example, consider the ordered pairs $(2, 2), (3, 4) \in \mathbb{R}^2$. The binary operation of vector addition in $\mathbb{R}^2$ assigns $(2, 2)$ and $(3, 4)$ to the vector $(5, 6)$ (another vector in $\mathbb{R}^2$).

The binary function of a vector space is called **scalar multiplication**. It takes some scalar in the field the vector space is defined over $k \in F$ and an element of the vector space $\mathbf{v} \in V$ and returns another vector in V written as $k\mathbf{v}$. For example, consider the vector $(2, 2) \in \mathbb{R}^2$ and the scalar $3 \in \mathbb{R}$. Then, the binary function of scalar multiplication takes in the inputs of $(2, 2)$ and 3 and returns the vector $(6, 6) \in \mathbb{R}^2$.

Note that both the binary operation and binary function exhibit **closure**. That is, their output is also an element of the vector space they act on.

It is important to note that our understanding of vectors as "representing a direction and magnitude" is only an **interpretation**. Vectors do **NOT** inherently "mean" this. Instead, we have chosen to interpret $(3, 2) \in \mathbb{R}^2$ as some point on the Cartesian plane in some instances, and we have chosen for it to represent some displacement from an origin O in others. In other contexts, they can mean other things.

One example you are probably familiar with is RGB values. An RGB value like $(255, 0, 0)$ is a 3-tuple, and hence would technically be an element of the vector space $\mathbb{R}^3$. However, we don't think of RGB values in the same way we normally think about vectors representing a magnitude and direction. **Context matters**.

Section 1.1 Vector Space Axioms

For a set of objects equipped with vector addition, scalar multiplication, and a zero element to constitute a vector space, it must satisfy the following 8 axioms (a statement that is taken to be true):

- **Commutativity of vector addition** - For all $\mathbf{v}$ and $\mathbf{w}$ in V, we have $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
- **Associativity of vector addition** - For all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V, we have $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

- **0 is a left identity for vector addition** - For all $\mathbf{v} \in V$, we have $\mathbf{0} + \mathbf{v} = \mathbf{v}$

Note: $\mathbf{v} + \mathbf{0} = \mathbf{v}$ is not an axiom. However, it can be proved through the first axiom and this third axiom.

- **Existence of additive inverses** - For each $\mathbf{v} \in V$ there exists some $\mathbf{w} \in V$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$
- **1 is an identity for scalar multiplication** - For all $\mathbf{v} \in V$ we have $1\mathbf{v} = \mathbf{v}$.
- **Associativity of scalar multiplication** - For all $k_1, k_2 \in F$ and all $\mathbf{v} \in V$ we have $(k_1 k_2)\mathbf{v} = k_1(k_2\mathbf{v})$
- **Distributivity of scalar multplication over vector addition** - For all $k \in F$ and $\mathbf{v}, \mathbf{w} \in V$ we have $k(\mathbf{v} + \mathbf{w}) = k\mathbf{v} + k\mathbf{w}$
- **Distributivity of scalar multiplication over field addition** - For all $k_1, k_2 \in F$ and all $\mathbf{v} \in V$ we have $(k_1 + k_2)\mathbf{v} = k_1\mathbf{v} + k_2\mathbf{v}$